Chapter 9

Lie algebra

• An (real) ***algebra*** is a (real) vector space equipped with a bilinear operation (product) under which the algebra is closed, i.e., for an algebra $\mathscr{A}$

i) $x \bullet y \in \mathscr{A} \qquad \forall x, y \in \mathscr{A}$

ii) $(\lambda x + \mu y) \bullet z = \lambda x \bullet z + \mu y \bullet z$
$x \bullet (\lambda y + \mu z) = \lambda x \bullet y + \mu x \bullet z \qquad \forall x, y, z \in \mathscr{A}, \quad \lambda, \mu \in \mathbb{R}\,.$

If λ, μ are complex numbers and $\mathscr{A}$ is a complex vector space, we get a ***complex algebra***. □

• A ***Lie algebra*** is an algebra in which the operation is

i) antisymmetric, $x \bullet y = -y \bullet x$, and

ii) satisfies the ***Jacobi identity***,

$$(x \bullet y) \bullet z + (y \bullet z) \bullet x + (z \bullet x) \bullet y = 0\,. \tag{9.1}$$

□

The Jacobi identity is not really an identity — it does not hold for an arbitrary algebra — but it must be satisfied by an algebra for it to be called a Lie algebra.

Example:

i) The space $\mathscr{M}_n = \{\text{all}\, n \times n \text{ matrices}\}$ under matrix multiplication, $A \bullet B = AB$. This is an ***associative algebra*** since matrix multiplication is associative, $(AB)C = A(BC)$.

ii) The same space $\mathscr{M}_n$ of all $n \times n$ matrices as above, but now with matrix commutator as the product,

$$A \bullet B = [A\,, B] = AB - BA\,. \tag{9.2}$$

This product is antisymmetric and satisfies Jacobi identity, so $\mathscr{M}_n$ with this product is a Lie algebra.

iii) The **angular momentum algebra** in quantum mechanics. If L_i are the angular momentum operators with $[L_i\,, L_j] = \mathrm{i}\epsilon_{ijk}L_k$, we can write the elements of this algebra as

$$\mathbb{L} = \left\{ a = \sum_i \zeta_i L_i | \zeta_i \in \mathbb{C} \right\} \tag{9.3}$$

If $a = \sum a_i L_i$ and $b = \sum b_i L_i$, their product is

$$a \bullet b \equiv [a\,, b] = \sum a_i b_j [L_i\,, L_j] = \mathrm{i} \sum \epsilon_{ijk} a_i b_j L_k\,. \tag{9.4}$$

This is a Lie algebra because it $[a\,, a] = 0$ and the Jacobi identity is satisfied.

iv) The **Poisson bracket algebra** of a classical dynamical system consists of functions on the phase space, with the product defined by the Poisson bracket,

$$f \bullet g = [f\,, g]_{P.B.}\,. \tag{9.5}$$

This is a Lie algebra. As a vector space it is infinite-dimensional.

v) Vector fields on a manifold form a real Lie algebra under the commutator bracket, since the Jacobi identity is a genuine identity, i.e. automatically satisfied, as we have seen in the previous chapter. This algebra is infinite-dimensional. (It can be thought of as the Lie algebra of the group of diffeomorphisms, $Diff(\mathcal{M})$) .