Chapter 8

Lie brackets

A vector field v is a linear map $C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$ since it is basically a derivation at each point, $v : f \mapsto v(f)$. In other words, given a smooth function f , $v(f)$ is a smooth function on $\mathcal{M}$. Suppose we consider two vector fields u , v . Then $u(v(f))$ is also a smooth function, linear in f . But is $uv \equiv u \circ v$ a vector field? To find out, we consider

$$\begin{aligned} u(v(fg)) &= u(fv(g) + v(f)g) \\ &= u(f)v(g) + fu(v(g)) + u(v(f))g + v(f)u(g) \,. \end{aligned} \tag{8.1}$$

We reorder the terms to write this as

$$uv(fg) = fuv(g) + uv(f)g + u(f)v(g) + v(f)u(g) \,, \tag{8.2}$$

so Leibniz rule is not satisfied by uv . But if we also consider the combination vu , we get

$$vu(fg) = f(vu(g) + vu(f)g + v(f)u(g) + u(f)v(g) \,. \tag{8.3}$$

Thus

$$(uv - vu)(fg) = f(uv - vu)(g) + (uv - vu)(f)g \,, \tag{8.4}$$

which means that the combination

$$[u\,, v] := uv - vu \tag{8.5}$$

is a vector field on $\mathcal{M}$, with the product uv signifying successive operation on any smooth function on $\mathcal{M}$.

• This combination is called the **commutator** or **Lie bracket** of the vector fields u and v. □

In any chart around the point $P \in \mathcal{M}$, we can write a vector field in local coordinates

$$v(f) = v^i \frac{\partial f}{\partial x^i}, \tag{8.6}$$

so that

$$\begin{aligned} u(v(f)) &= u^j \frac{\partial}{\partial x^j}\left(v^i \frac{\partial f}{\partial x^i}\right) \\ &= u^j \frac{\partial v^i}{\partial x^j}\frac{\partial f}{\partial x^i} + u^j v^i \frac{\partial^2 f}{\partial x^j x^i}, \\ v(u(f)) &= v^j \frac{\partial u^i}{\partial x^j}\frac{\partial f}{\partial x^i} + u^j v^i \frac{\partial^2 f}{\partial x^j x^i}. \end{aligned} \tag{8.7}$$

Subtracting, we get

$$u(v(f)) - v(u(f)) = u^j \frac{\partial v^i}{\partial x^j}\frac{\partial f}{\partial x^i} - v^j \frac{\partial u^i}{\partial x^j}\frac{\partial f}{\partial x^i}, \tag{8.8}$$

from which we can read off the components of the commutator,

$$[u\,, v]^i = u^j \frac{\partial v^i}{\partial x^j} - v^j \frac{\partial u^i}{\partial x^j} \tag{8.9}$$

• The commutator is antisymmetric, $[u\,, v] = -[v\,, u]$, and satisfies the **Jacobi identity**

$$[[u\,, v]\,, w] + [[v\,, w]\,, u] + [[w\,, u]\,, v] = 0\,. \tag{8.10}$$

The commutator is useful for the following reason: Once we have a chart, we can use $\left\{\dfrac{\partial}{\partial x^i}\right\}$ as a basis for vector fields in a neighbourhood.

Any set of n linearly independent vector fields may be chosen as a basis, but they need not form a coordinate system. In a coordinate system,

$$\left[\frac{\partial}{\partial x^i}\,, \frac{\partial}{\partial x^j}\right] = 0\,, \tag{8.11}$$

because partial derivatives commute. So n vector fields will form a coordinate system only if they commute, i.e., have vanishing commutators with one another. Then the coordinate lines are the integral curves of the vector fields. For analytic manifolds, this condition is sufficient as well.

A simple example is the polar coordinate system in $\mathbb{R}^2$. The unit vectors are

$$\begin{aligned} e_r &= e_x \cos\theta + e_y \sin\theta \\ e_\theta &= -e_x \sin\theta + e_y \cos\theta\,, \end{aligned} \tag{8.12}$$

with $e_x = \dfrac{\partial}{\partial x}$ and $e_y = \dfrac{\partial}{\partial y}$ being the Cartesian coordinate basis vectors, and

$$\cos\theta = \frac{x}{r}\,, \qquad \sin\theta = \frac{y}{r}\,, \qquad r = \sqrt{x^2 + y^2} \tag{8.13}$$

Using these expressions, it is easy to show that $[e_r\,, e_\theta] \neq 0$, so $\{e_r, e_\theta\}$ do not form a coordinate basis.