Chapter 7

Pull back and push forward

Two important concepts are those of pull back (or pull-back or pull-back) and push forward (or push-forward or pushforward) of maps between manifolds.

• Given manifolds $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ and maps $f : \mathcal{M}_1 \to \mathcal{M}_2\,, g : \mathcal{M}_2 \to \mathcal{M}_3\,$, the **pullback** of g under f is the map $f^*g : \mathcal{M}_1 \to \mathcal{M}_3$ defined by

$$f^*g = g \circ f\,. \tag{7.1}$$

□ So in particular, if $\mathcal{M}_1$ and $\mathcal{M}_2$ are two manifolds with a map $f : \mathcal{M}_1 \to \mathcal{M}_2$ and $g : \mathcal{M}_2 \to \mathbb{R}$ is a function on $\mathcal{M}_2\,$, the pullback of g under f is a function on $\mathcal{M}_1\,$,

$$f^*g = g \circ f\,. \tag{7.2}$$

While this looks utterly trivial at this point, this concept will become increasingly useful later on.

• Given two manifolds $\mathcal{M}_1$ and $\mathcal{M}_2$ with a smooth map $f : \mathcal{M}_1 \to \mathcal{M}_2, P \mapsto Q$ the **pushforward** of a vector $v \in T_P\mathcal{M}_1$ is a vector $f_*v \in T_Q\mathcal{M}_2$ defined by

$$f_*v(g) = v(g \circ f) \tag{7.3}$$

for all smooth functions $g : \mathcal{M}_2 \to \mathbb{R}\,$. □

Thus we can write

$$f_*v(g) = v(f^*g)\,. \tag{7.4}$$

The pushforward is linear,

$$f_*(v_1 + v_2) = f_* v_1 + f_* v_2 \tag{7.5}$$
$$f_*(\lambda v) = \lambda f_* v\,. \tag{7.6}$$

And if $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$ are manifolds with maps $f : \mathcal{M}_1 \to \mathcal{M}_2\,, g : \mathcal{M}_2 \to \mathcal{M}_3$, it follows that

$$(g \circ f)_* = g_* f_*\,, \qquad i.e.$$
$$(g \circ f)_* v = g_* f_* v \qquad \forall v \in T_P\mathcal{M}_1\,. \tag{7.7}$$

Remember that we can think of a vector v as an equivalence class of curves $[\gamma]$. The pushforward of an equivalence class of curves is

$$f_* v = f_*[\gamma] = [f \circ \gamma] \tag{7.8}$$

Note that for this pushforward to be defined, we do not need the original maps to be 1-1 or onto. In particular, the two manifolds may have different dimensions.

Suppose $\mathcal{M}_1$ and $\mathcal{M}_2$ are two manifolds with dimension m and n respectively. So in the respective tangent spaces $T_P\mathcal{M}_1$ and $T_Q\mathcal{M}_2$ are also of dimension m and n respectively. So for a map $f : \mathcal{M}_1 \to \mathcal{M}_2, P \mapsto Q$, the pushforward f_* will not have an inverse if $m \neq n$.

Let us find the components of the pushforward $f_* v$ in terms of the components of v for any vector v. Let us in fact consider, given charts $\varphi : P \mapsto (x^1, \cdots, x^m)\,, \psi : Q \mapsto (y^1, \cdots, y^n)$ the pushforward of the basis vectors.

For the basis vector $\left(\frac{\partial}{\partial x^i}\right)_P$, we want the pushforward $f_*\left(\frac{\partial}{\partial x^i}\right)_P$, which is a vector in $T_Q\mathcal{M}_2$, so we can expand it in the basis $\left(\frac{\partial}{\partial y^i}\right)_Q$,

$$f_*\left(\frac{\partial}{\partial x^i}\right)_P = \left(f_*\left(\frac{\partial}{\partial x^i}\right)_P\right)^\mu \left(\frac{\partial}{\partial y^\mu}\right)_Q \tag{7.9}$$

In any coordinate basis, the components of a vector are given by the action of the vector on the coordinates as in Chap. 4,

$$v_P^\mu = v_P(y^\mu) \tag{7.10}$$

Thus we can write

$$\left(f_*\left(\frac{\partial}{\partial x^i}\right)_P\right)^\mu = f_*\left(\frac{\partial}{\partial x^i}\right)_P (y^\mu) \tag{7.11}$$

But

$$f_* v(g) = v(g \circ f)\,, \tag{7.12}$$

so

$$f_* \left(\frac{\partial}{\partial x^i}\right)_P (y^\mu) = \left(\frac{\partial}{\partial x^i}\right)_P (y^\mu \circ f)\;. \tag{7.13}$$

But $y^\mu \circ f$ are the coordinate functions of the map f, i.e., coordinates around the point $f(P) = Q$. So we can write $y^\mu \circ f$ as $y^\mu(\boldsymbol{x})$, which is what we understand by this. Thus

$$\left(f_* \left(\frac{\partial}{\partial x^i}\right)_P\right)^\mu = \left(\frac{\partial}{\partial x^i}\right)_P (y^\mu \circ f) = \left.\frac{\partial y^\mu(\boldsymbol{x})}{\partial x^i}\right|_P . \tag{7.14}$$

Because we are talking about derivatives of coordinates, these are actually done in charts around P and $Q = f(P)$, so the chart maps are hidden in this equation.

• The right hand side is called the **Jacobian matrix** (of $y^\mu(\boldsymbol{x}) = y^\mu \circ f$ with respect to x^i). Note that since m and n may be unequal, this matrix need not be invertible and a determinant may not be defined for it. □

For the basis vectors, we can then write

$$f_* \left(\frac{\partial}{\partial x^i}\right)_P = \left.\frac{\partial y^\mu(\boldsymbol{x})}{\partial x^i}\right|_P \left(\frac{\partial}{\partial y^\mu}\right)_{f(P)} \tag{7.15}$$

Since f_* is linear, we can use this to find the components of $(f_* v)_Q$ for any vector v_P,

$$\begin{aligned} f_* v_P &= f_* \left[v_P^i \left(\frac{\partial}{\partial x^i}\right)_P \right] \\ &= v_P^i f_* \left(\frac{\partial}{\partial x^i}\right)_P \\ &= v_P^i \left.\frac{\partial y^\mu(\boldsymbol{x})}{\partial x^i}\right|_P \left(\frac{\partial}{\partial y^\mu}\right)_{f(P)} \end{aligned} \tag{7.16}$$

$$\Rightarrow \qquad (f_* v_P)^\mu = v_P^i \left.\frac{\partial y^\mu(\boldsymbol{x})}{\partial x^i}\right|_P . \tag{7.17}$$

Note that since f_* is linear, we know that the components of $f_* v$ should be linear combinations of the components of v, so we can already guess that $(f_* v_P)^\mu = A^\mu_i v^i_P$ for some matrix A^μ_i . The matrix is made of first derivatives because vectors are first derivatives.

Another example of the pushforward map is the following. Remember that tangent vectors are derivatives along curves. Suppose $v_P \in T_P\mathcal{M}$ is the derivative along γ . Since $\gamma : I \to \mathcal{M}$ is a map, we can consider pushforwards under γ , of derivatives on I . Thus for $\gamma : I \to \mathcal{M}\,, t \mapsto \gamma(t) = P$, and for some $g : \mathcal{M} \to \mathbb{R}$,

$$\begin{aligned} \gamma_* \left(\frac{d}{dt}\right)_{t=0} g &= \frac{d}{dt}\,(g \circ \gamma)\big|_{t=0} \\ &= \dot{\gamma}_P(g)\big|_{t=0} = v_P(g)\,, \end{aligned} \tag{7.18}$$

so

$$\gamma_* \left(\frac{d}{dt}\right)_{t=0} = v_P \tag{7.19}$$

• We can use this to give another definition of integral curves. Suppose we have a vector field v on $\mathcal{M}$. Then the integral curve of v passing through $P \in \mathcal{M}$ is a curve $\gamma : t \mapsto \gamma(t)$ such that $\gamma(0) = P$ and

$$\gamma_* \left(\frac{d}{dt}\right)_t = v\big|_{\gamma(t)} \tag{7.20}$$

for all t in some interval containing P . □

Even though in order to define the pushforward of a vector v under a map f , we do not need f to be invertible, the pushforward of a vector field can be defined only if f is both one-to-one and onto.

If f is not one-to-one, different points P and P' may have the same image, $f(P) = Q = f(P')$. Then for the same vector field v we must have

$$f_* v\big|_Q = f_*(v_P) = f_*(v_{P'})\,, \tag{7.21}$$

which may not be true. And if $f : \mathcal{M} \to \mathcal{N}$ is not onto, $f_* v$ will be meaningless outside some region $f(\mathcal{M})$, so $f_* v$ will not be a vector field on $\mathcal{N}$.

If f is one-to-one and onto, it is a diffeomorphism, in which case vector fields can be pushed forward, by the rule

$$(f_* v)_{f(P)} = f_*\left(v_P\right)\,. \tag{7.22}$$