Chapter 6

Vector fields

• Consider the (disjoint) union of tangent spaces at all points,

$$T\mathcal{M} = \bigcup_{P\in\mathcal{M}} T_P\mathcal{M}\,. \tag{6.1}$$

This is called the **tangent bundle** of $\mathcal{M}$. □

• A **vector field** v chooses an element of $T_P\mathcal{M}$ for every P, i.e. $v : P \mapsto v(P) \equiv v_P \in T_P\mathcal{M}$. □

We will often write $v(f)|_P = v_P(f)$.

Given a chart, v has components v^i in the chart,

$$v_P = v^i \left(\frac{\partial}{\partial x^i}\right)_P\,, \qquad (v^i)_P = v_P(x^i)\,. \tag{6.2}$$

• The vector field v is **smooth** if the functions $v^i = v(x^i)$ are smooth for any chart (and thus for all charts). □

• A rule that selects a covector from $T^*_P\mathcal{M}$ for each P is called a **one−form** (often written as a **1−form**). □

• Given a smooth vector field v (actually C^1 is sufficient) we can define an **integral curve** of v, which is a curve γ in $\mathcal{M}$ such that $\dot\gamma(t)|_P = v_P$ at every $P \in \gamma$. (One curve need not pass through all $P \in \mathcal{M}$.) □

Suppose γ is an integral curve of a given vector field v, with $\gamma(0) = P$. Then in a chart containing P, we can write

$$\dot\gamma(t) = v \Rightarrow \frac{d}{dt}x^i(\gamma(t)) = v^i(\boldsymbol{x}(t))\,, \tag{6.3}$$

with initial condition $x^i(0) = x^i|_P$. This is a set of ordinary first order differential equations. If v^i are smooth, the theory of differential equations guarantees, at least for small t (i.e. locally), the existence of exactly one solution. The uniqueness of these solutions implies that the integral curves of a vector field do not cross.

One use of integral curves is that they can be thought of as coordinate lines. Given a smooth vector field v such that $v|_P \neq 0$, it is possible to define a coordinate system $\{x^i\}$ in a neighbourhood around P such that $v = \frac{\partial}{\partial x^i}$.

• A vector field v is said to be **complete** if at every point $P \in \mathcal{M}$ the integral curve the integral curve $\gamma(t)$ of v passing through P can be extended to all $t \in \mathbb{R}$. □

The tangent bundle $T\mathcal{M}$ is a product manifold, i.e., a point in $T\mathcal{M}$ is an ordered pair (P, v) where $P \in \mathcal{M}$ and $v \in T_P\mathcal{M}$. The topological structure and differential structure are given appropriately.

• The map $\pi : T\mathcal{M} \to \mathcal{M}, (P, v) \mapsto P$ (where $v \in T_P\mathcal{M}$) is called the **canonical projection** (or simply **projection**). □

• For each $P \in \mathcal{M}$, the pre-image $\pi^{-1}(P)$ is $T_P\mathcal{M}$. It is called the **fiber** over P. Then a vector field can be thought of as a **section** of the tangent bundle. □

Given a smooth vector field v, we can define an integral curve γ through any point P by $\dot{\gamma}(t) = v$, i.e.,

$$\frac{d}{dt}x^i(\gamma(t)) = v^i(\gamma(t)) \equiv v(x^i(\gamma(t))) , \tag{6.4}$$

$$\gamma(0) = P . \tag{6.5}$$

We could also choose $\gamma(t_0) = P$.

Then in any neighbourhood U of P we also have γ_Q , the integral curve through Q. So we can define a map $\phi : I \times U \to \mathcal{M}$ given by $\phi(t, Q) = \gamma_Q(t)$ where $\gamma_Q(t)$ satisfies

$$\frac{d}{dt}x^i(\gamma_Q(t)) = v(x^i(\gamma_Q(t))) , \tag{6.6}$$

$$\gamma_Q(0) = Q . \tag{6.7}$$

• This ϕ defines a map $\phi_t : U \to \mathcal{M}$ at each t by $\phi_t(Q) = \phi(t, Q) = \gamma_Q(t)$, i.e. for given t, ϕ_t takes a point by a parameter distance t along the curve $\gamma_Q(t)$. This ϕ_t is called the **local flow** of v. □

The local flow has the following properties:

$i)$ ϕ_0 is the identity map of U ;

ii) $\phi_s \circ \phi_t = \phi_{s+t}$ for all $s, t, s+t \in U$;

iii) each flow is a diffeomorphism with ${\phi_t}^{-1} = \phi_{-t}$.

The first property is obvious, while the second property follows from the uniqueness of integral curves, i.e. of solutions to first order differential equations. Then the integral curve passing through the point $\gamma_{_Q}(s)$ is the same as the integral curve passing through Q, so that moving a parameter distance t from $\gamma_{_Q}(s)$ finds the same point on $\mathcal{M}$ as by moving a parameter distance $s+t$ from $\gamma_{_Q}(0) \equiv Q$.

A vector field can also be thought of as a map from the space of differentiable functions to itself $v : C^\infty(\mathcal{M}) \to C^\infty(\mathcal{M})$, $f \mapsto v(f)$, with $v(f) : \mathcal{M} \to \mathbb{R}$, $P \mapsto v_{_P}(f)$. Often $v(f)$ is called the Lie derivative of f along v and denoted $\pounds_v f$.

The map $v : f \mapsto v(f)$ has the following properties:

$$v(f + \alpha g) = v(f) + \alpha v(g) \tag{6.8}$$
$$v(fg) = f v(g) + v(f) g \tag{6.9}$$
$$\forall f, g \in C^\infty(\mathcal{M}), \qquad \alpha \in \mathbb{R}$$

The set of all (real) vector fields $V(\mathcal{M})$ on a manifold $\mathcal{M}$ has the structure of a (real) vector space under vector addition defined by

$$(u + \alpha v)(f) = u(f) + \alpha v(f), \qquad u, v \in V(\mathcal{M}), \qquad \alpha \in \mathbb{R}\,. \tag{6.10}$$

It is possible to replace α by some function on $C^\infty(\mathcal{M})$. If u, v are vector fields on $\mathcal{M}$ and α is now a smooth function on $\mathcal{M}$, define $u + \alpha v$ by

$$(u + \alpha v)_{_P}(f) = u_{_P}(f) + \alpha(P) v_{_P}(f) \qquad \forall f \in C^\infty(\mathcal{M}), \qquad P \in \mathcal{M}\,. \tag{6.11}$$

This looks like a vector space but actually it is what is called a **module**.

• A **ring** R is a set or space with addition and multiplication defined on it, satisfying $(xy)z = x(yz)$, $x(y+z) = xy + xz$, $(x+y)z = xz + yz$, and two special elements 0 and 1, the additive and multiplicative identity elements, $0 + x = x + 0 = x$, $1x = x1 = x$. A **module** X is an Abelian group under addition, with scalar multiplication by elements of a ring defined on it.

• A module becomes a ***vector space*** when this ring is a ***commu tative division ring***, i.e. when the ring multiplication is commutative, $xy = yx$, and an inverse exists for every element except 0. Given a smooth function α, in general $\alpha^{-1} \notin C^\infty(\mathcal{M})$, so the space of vector fields on $\mathcal{M}$ is in general a module, not a vector space.

Given a vector field v, in an open neighbourhood of some $P \in \mathcal{M}$ and in a chart, and for any $f \in C^\infty(\mathcal{M})$, we have

$$v(f)\Big|_P = v_P(f) = v_P^i \left(\frac{\partial f}{\partial x^i}\right)_P , \qquad \text{where} \quad v_P^i = v_P(x^i)\,. \tag{6.12}$$

Thus we can write

$$v = v^i \frac{\partial}{\partial x^i} \qquad \text{with} \quad v^i = v(x^i)\,, \tag{6.13}$$

as an obvious generalization of vector space expansion to the module $V(\mathcal{M})$.

The v^i are now the ***components*** of the vector field v, and $\frac{\partial}{\partial x^i}$ are now vector fields, which we will call the ***coordinate vector fields***. Note that this is correct only in some open neighbourhood on which a chart can be defined. In particular, it may not be possible in general to define the coordinate vector fields globally, i.e. everywhere on $\mathcal{M}$, and thus the components v^i may not be defined globally either.