Chapter 24

Connections

There is no canonical way to differentiate sections of a fiber bundle. That is to say, no unique derivative arises from the definition of a bundle. Let us see why. The usual derivative of a function on $\mathbb{R}$ is of the form

$$f'(x) = \lim_{\epsilon \to 0} \frac{f(x+\epsilon) - f(x)}{\epsilon}\,. \tag{24.1}$$

But a section of a bundle assigns an element of the fiber at any point $P \in \mathcal{M}$ to the base point P. Each fiber is isomorphic to the standard fiber but the isomorphism is not canonical or unique. So there is no unique way of adding or subtracting points on different fibers. Thus there are many ways of differentiating sections of fiber bundles.

• Each way of taking a derivative, i.e. of comparing, is called a ***connection***. Let us consider a bundle $\pi : E \to B$, where $\Gamma(E)$ is the space of all sections. Then ***a connection*** D on B assigns to every vector field v on B a map $D_v : \Gamma(E) \to \Gamma(E)$ satisfying

$$\begin{aligned} D_v(s + \alpha t) &= D_v s + \alpha\, D_v t \\ D_v(fs) &= v(f)s + f D_v s \\ D_{v+fw}(s) &= D_v s + f D_w s\,, \end{aligned} \tag{24.2}$$

where s, t are sections of the bundle, $s, t \in \Gamma(E)$, v, w are vector fields on B, $f \in C^\infty(B)$ and α is a real number (or complex, depending on what the manifold is). □

Note that this is ***a connection***, not some unique connection. In other words, we need to choose a connection before we can talk about it. In what follows, whenever we refer to a connection D, we mean that we have chose a connection D and that is what we are discussing.

- We call $D_v s$ the **covariant derivative** of s. □

To be specific, let us consider the bundle to be a vector bundle on a manifold $\mathcal{M}$, and try to understand the meaning of D by going to a chart. Consider coordinates x^μ in an open set $U \subset \mathcal{M}$, with ∂_μ the coordinate basis vector fields. Write $D_\mu = D_{\partial_\mu}$. Also choose a basis for sections, which is like a basis for the fiber (vector space) at each point of $\mathcal{M}$, i.e. like a set of basis vector fields, but the vectors are not along $\mathcal{M}$, but along the fiber at $\mathcal{M}$.

Call this basis $\{e_i\}$, then $\{e_i(x)\}$ is a basis for the fiber at $P \in \mathcal{M}$, with $\{x\}$ being the set of coordinates at P. Any element of $V \simeq F_x$ can be written uniquely as a linear combination of $e_i(x)$. But then $D_\mu e_j$ can be expressed uniquely as a linear combination of the e_i,

$$D_\mu e_j = A_{\mu j}{}^i e_i \,. \tag{24.3}$$

- These $A^i_{\mu j}$ are called **components** of the **vector potential** or the **connection one-form**. □

Given a section $s = s^i e_i$ with $s^i \in C^\infty(\mathcal{M})$, we can write

$$D_v s = D_{v^\mu \partial_\mu} s = v^\mu D_\mu s \,. \tag{24.4}$$

Also,

$$\begin{aligned} D_\mu s = D_\mu(s^i e_i) &= \left(\partial_\mu s^i\right) e_i + s^i D_\mu e_i \\ &= \left(\partial_\mu s^i\right) e_i + s^i A_{\mu i}{}^j e_j \\ &= \left(\partial_\mu s^i + A_{\mu j}{}^i s^j\right) e_i \,, \end{aligned} \tag{24.5}$$

so writing $D_\mu s = (D_\mu s)^i e_i$, we can say

$$(D_\mu s)^i = \partial_\mu s^i + A_{\mu j}{}^i s^j \,. \tag{24.6}$$

We have considered connections on an associated vector bundle, which may be a principal fiber bundle such as a frame bundle. So we should be able to talk about gauge transformations.

Remember that a gauge transformation is a linear map $T : E \to E$, $(x, v) \mapsto (x, gv)$ for some $g \in G$ and for all $v \in V \simeq F_x$. Let us apply this idea to the section s. We claim that given a connection D, there is a connection D' on E such that

$$D'(g\phi) = g D_v \phi \,, \tag{24.7}$$

where v is a vector field on $\mathcal{M}$ and $\phi \in \Gamma(E)$ (i.e. $\phi = s$).

Let us first check if the definition makes sense. Since $g(x) \in G$ for all $x \in \mathcal{M}$, we know that $g^{-1}(x)$ exists for all x. So

$$\begin{aligned} D'_v(\phi) &= D'_v\left(gg^{-1}\phi\right) \\ &= gD_v\left(g^{-1}\phi\right) , \end{aligned} \tag{24.8}$$

and thus D' is defined on all ϕ for which $D_v\phi$ is defined, i.e. D' exists because D does. We have of course assumed that $g(x)$ is differentiable as a function of x.

Let us now check that D' is a connection according to our definitions. D' is linear since

$$\begin{aligned} D'_v\left(\phi_1 + \alpha\phi_2\right) &= gD_v\left(g^{-1}\left(\phi_1 + \alpha\phi_2\right)\right) \\ &= gD_v\left(g^{-1}\phi_1\right) + \alpha g D_v\left(g^{-1}\phi_2\right) \\ &= D'_v\phi_1 + \alpha D'_v\phi_2 \,. \end{aligned} \tag{24.9}$$

And it satisfies Leibniz rule because

$$\begin{aligned} D'_v(f\phi) &= gD_v\left(g^{-1}f\phi\right) \\ &= gD_v\left(f\left(g^{-1}\phi\right)\right) \\ &= gv(f)g^{-1}\phi + gfD_v\left(g^{-1}\phi\right) \\ &= v(f)\phi + fgD_v\left(g^{-1}\phi\right) \\ &= v(f)\phi + fD'_v(\phi) \,. \end{aligned} \tag{24.10}$$

Similarly,

$$\begin{aligned} D'_{v+\alpha w}\phi &= gD_{v+\alpha w}\left(g^{-1}\phi\right) \\ &= g\left(D_v\left(g^{-1}\phi\right) + \alpha D_w\left(g^{-1}\phi\right)\right) \\ &= gD_v(g^{-1}\phi) + \alpha g D_w\left(g^{-1}\phi\right) \\ &= D'_v\phi + \alpha D'_w\phi \,. \end{aligned} \tag{24.11}$$

So D' is a connection, i.e. there is a connection D' satisfying $D'_v(g\phi) = g\left(D_v\phi\right)$.

Since ϕ is a section, i.e. $\phi \in \Gamma(E)$, so is $g^{-1}\phi$, and thus $D_v\left(g^{-1}\phi\right) \in \Gamma(E)$ and also $gD_v\left(g^{-1}\phi\right) \in \Gamma(E)$. Therefore, D'_v maps sections to sections, $D'_v : \Gamma(E) \to \Gamma(E)$. This completes the definition of the gauge transformation of the connection. We can now write

$$D'_\mu\phi = \left(\partial_\mu\phi^i + A'^{\,i}_{\mu\ j}\,\phi^j\right) e_i \,. \tag{24.12}$$

Using the dual space, let us write

$$\begin{aligned} A_\mu &= A_{\mu}{}^{i}{}_{j}\, e_i \otimes \theta^j \,, \\ A'_\mu &= A'_{\mu}{}^{i}{}_{j}\, e_i \otimes \theta^j \,, \end{aligned} \tag{24.13}$$

where $\{\theta^i\}$ is the dual basis to $\{e_i\}$. The gauge transformation is then given by

$$\begin{aligned} D'_v\phi &= gD_v\left(g^{-1}\phi\right) \\ \Rightarrow \qquad D'_\mu\phi &= gD_\mu\left(g^{-1}\phi\right) \\ \Rightarrow \qquad \left(\partial_\mu\phi^i + A'_{\mu}{}^{i}{}_{j}\right) e_i &= g\left[\partial_\mu\left(g^{-1}\phi\right)^i + A_{\mu}{}^{i}{}_{j}\left(g^{-1}\phi\right)^j\right] e_i \,, \end{aligned} \tag{24.14}$$

where, as always, the g's are in some appropriate representation of G. Then we can write the right hand side as

$$\begin{aligned} &\left[\partial_\mu\phi^i + \left(g\partial_\mu g^{-1}\right)^i_j\phi^j + \left(gA_\mu g^{-1}\right)^i_j\phi^j\right] e_i \\ &\quad = \left[\partial_\mu\phi^i + \left(g\partial_\mu g^{-1} + gA_\mu g^{-1}\right)^i_j\phi^j\right] e_i \,. \end{aligned} \tag{24.15}$$

From this we can read off

$$A'_\mu = gA_\mu g^{-1} + g\partial_\mu g^{-1} \,. \tag{24.16}$$

• A connection which transforms like this is also called a *G*-**connection**. □

Example: Consider $G = U(1)$. Suppose E is a trivial complex line bundle over $\mathcal{M}$, i.e. $E = \mathcal{M}\times\mathbb{C}$, so that the fiber over any point $p\in\mathcal{M}$ is $\mathbb{C}$. A connection D on E may be written as $D_\mu = \partial_\mu + A_\mu$. We can make E into a $U(1)$ bundle by thinking of the fiber $\mathbb{C}$ as the fundamental representation space of $U(1)$. Then sections are complex functions, and a gauge transformation is multiplication by a phase. □