Chapter 21

Tangent space at the identity

A point on the Lie group is a group element. So a vector field on the Lie group selects a vector at each $g \in G$. Since left and right translations are diffeomorphisms, we can consider the pushforwards due to them.

• A **left-invariant vector field** X is invariant under left ttranslations, i.e.,

$$X = l_{g*}(X) \qquad \forall g \in G\,. \tag{21.1}$$

In other words, the vector (field) at g' is pushed forward by l_g to the same vector (field) at $l_g(g')$:

$$l_{g*}(X_{g'}) = X_{gg'} \qquad \forall g, g' \in G\,. \tag{21.2}$$

• Similarly, a **right-invariant vector field** X is defined by

$$\begin{aligned} X &= r_{g*}(X) \qquad \forall g \in G\,, \\ \text{i.}e. \quad r_{g*}(X_{g'}) &= X_{g'g} \qquad \forall g, g' \in G\,. \end{aligned} \tag{21.3}$$

A left or right invarian vector field has the important property that it is completely determined by its value at the identity element e of the Lie group, since

$$l_{g*}(X_e) = X_g \qquad \forall g \in G\,, \tag{21.4}$$

and similarly for right-invariant vector fields.

Write the set of all left-invariant vector fields on G as $L(G)$. Since the push-forward is linear, we get

$$l_{g*}(aX + Y) = a l_{g*}X + l_{g*}Y\,, \tag{21.5}$$

so that if both X and Y are left-invariant,

$$l_{g*}(aX + Y) = aX + Y\,, \tag{21.6}$$

so the set of left-invariant vector fields form a real vector space.

We also know that push-forwards leave the Lie algebra invariant, i.e., for l_{g*},

$$[l_{g*}X\,, l_{g*}Y] = l_{g*}[X\,, Y]\,. \tag{21.7}$$

Thus if $X, Y \in L(G)$,

$$l_{g*}[X\,, Y] = [l_{g*}X\,, l_{g*}Y] = [X\,, Y]\,, \tag{21.8}$$

so $[X\,, Y] \in L(G)$. Thus the set of all left-invariant vector fields on G forms a Lie algebra.

- This $L(G)$ is called the **Lie algebra of** G. □

The dimension of this Lie algebra is the same as that of G because of the

Theorem: $L(G)$ as a real vector space is isomorphic to the tangent space T_eG to G at the identity of G.

Proof: We will show that left translation leads to an isomorphism.

For $X \in T_eG$, define the vector field L^X on G by

$$L^X\big|_g \equiv L^X_g := l_{g*}X \qquad\qquad \forall g \in G \tag{21.9}$$

Then for all $g, g' \in G$,

$$l'_g{*}(L^X_g) = l_{g'*}(l_{g*}X) = l_{g'g*}X = L^X_{g'g}\,. \tag{21.10}$$

Note that for two diffeomorphisms φ_1, φ_2, we can write

$$\begin{aligned} (\varphi_{1*}(\varphi_{2*}v))(f) &= (\varphi_{2*}v)(f \circ \varphi_1) \\ &= v(f \circ \varphi_1 \circ \varphi_2) \\ &= ((\varphi_1 \circ \varphi_2)_*v)(f) \\ \Rightarrow \qquad \varphi_{1*}(\varphi_{2*}v) &= (\varphi_1 \circ \varphi_2)_*v \end{aligned} \tag{21.11}$$

Since left translation is a diffeomorphism,

$$l_{g'*}(l_{g*}X) = (l_{g'} \circ l_g)_*X = (l_{g'g*})X \tag{21.12}$$

So it follows that L^X is a left-invariant vector field, and we have a map $T_eG \to L(G)$. Since the pushforward is a linear map, so is the map $X \to L^X$. We need to prove that this map is 1-1 and onto.

If $L^X = L^Y$, we have

$$L^X_g = L^Y_g \qquad \forall g \in G\,, \tag{21.13}$$

so

$$l_{g^{-1}*}L^X_g = l_{g^{-1}*}L^Y_g \qquad \Rightarrow \qquad X = Y \quad (\in T_eG)\,. \tag{21.14}$$

So the map $X \to L^X$ is 1-1.

Now given L^X , define $X_e \in T_eG$ by

$$X_e = l_{g^{-1}*}L^X_g \qquad \text{for any } g \in G\,. \tag{21.15}$$

We can also write

$$X_e = L^X_e\,. \tag{21.16}$$

Then

$$l_{g*}X_e = l_{g*}l_{g^{-1}*}L^X_g = L^X_g\,. \tag{21.17}$$

So the map $X \mapsto L^X$ is onto.

Then we can define a Lie bracket on T_eG by

$$[u\,, v] = [L^u\,, L^v]\big|_e\,. \tag{21.18}$$

The Lie algebra of vectors in T_eG based on this bracket is thus the Lie algebra of the group G . It follows that

$$\dim L(G) = \dim T_eG = \dim G\,. \tag{21.19}$$

Note that since commutators are defined for vector fields and not vectors, the Lie bracket on T_eG has to be defined using the commutator of left-invariant vector fields on G and the isomorphism $T_eG \leftrightarrow L(G)$.

• If for an n-dimensional Lie group G , $\{t_1\,, \cdots, t_n\}$ is a set of basis vectors on $T_eG \simeq L(G)$, the Lie bracket of any pair of these vectors must be a linear combination of them, so

$$[t_i\,, t_j] = \sum_k C^k_{ij}\, t_k \tag{21.20}$$

for some set of real numbers C^k_{ij}. These numbers are known as the **structure constants** of the Lie group or algebra. □

Since $L(G)$ is a Lie algebra, with the Lie bracket as the product, the Lie bracket is antisymmetric,

$$\begin{aligned} [t_i\,,t_j] &= [t_j\,,t_i] \\ \Rightarrow \qquad \sum_k C^k_{ij}\,t_k &= \sum_k C^k_{ji} t_k \\ \Rightarrow \qquad C^k_{ij} &= C^k_{ji}\,, \end{aligned} \tag{21.21}$$

and the structure constants satisfy the Jacobi identity

$$\begin{aligned} [t_i\,,[t_j\,,t_k]] + [t_j\,,[t_k\,,t_i]] + [t_k\,,[t_i\,,t_j]] &= 0 \\ \Rightarrow \qquad C^l_{ij}\,C^m_{kl} + C^l_{jk}\,C^m_{il} + C^l_{ki}\,C^m_{jl} &= 0\,. \end{aligned} \tag{21.22}$$

A similar construction can be done using a set of right-invariant vector fields defined by

$$R^X_g := r_{g*}X \qquad \text{for } X \in T_eG \tag{21.23}$$

and its 'inverse' $X_e = r_{g^{-1}*}R^X_g$.