Chapter 18

Maxwell equations

We will now consider a particular example in physics where differential forms are useful. The Maxwell equations of electrodynamics are, with $c = 1$,

$$\nabla \cdot \boldsymbol{E} = \rho \tag{18.1}$$

$$\nabla \times \boldsymbol{B} - \frac{\partial \boldsymbol{E}}{\partial t} = \boldsymbol{j} \tag{18.2}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{18.3}$$

$$\nabla \times \boldsymbol{E} + \frac{\partial \boldsymbol{B}}{\partial t} = 0\,. \tag{18.4}$$

The electric and magnetic fields are all vectors in three dimensions, but these equations are Lorentz-invariant. We will write these equations in terms of differential forms.

Consider $\mathbb{R}^4$ with Minkowski metric $g_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. For the magnetic field define a 2-form

$$B = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy\,. \tag{18.5}$$

For the electric field define a 1-form

$$E = E_x dx + E_y dy + E_z dz\,. \tag{18.6}$$

Combine these two into a 2-form $F = B + E \wedge dt$. Let us calculate $dF = d(B + E \wedge dt) = dB + dE \wedge dt$. As usual, We will write $1, 2, 3$

for the component labels x, y, z.

$$\begin{aligned}
dB &= d(B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy) \\
&= \partial_t B_1 dt \wedge dy \wedge dz + \partial_1 B_1 dx \wedge dy \wedge dz \\
&\quad + \partial_t B_2 dt \wedge dz \wedge dx + \partial_2 B_2 dy \wedge dz \wedge dx \\
&\quad + \partial_t B_3 dt \wedge dx \wedge dy + \partial_3 B_3 dz \wedge dx \wedge dy\,. \qquad (18.7)
\end{aligned}$$

And

$$\begin{aligned}
d(E \wedge dt) &= d(E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt) \\
&= \partial_2 E_1 dy \wedge dx \wedge dt + \partial_3 E_1 dz \wedge dx \wedge dt \\
&\quad + \partial_1 E_2 dx \wedge dy \wedge dt + \partial_3 E_2 dz \wedge dy \wedge dt \\
&\quad + \partial_1 E_3 dx \wedge dz \wedge dt + \partial_2 E_3 dy \wedge dz \wedge dt\,. \qquad (18.8)
\end{aligned}$$

Thus, remembering that the wedge product changes sign under each exchange, we can combine these two to get

$$\begin{aligned}
dF &= (\partial_t B_1 + \partial_2 E_3 - \partial_3 E_2)\; dt \wedge dy \wedge dz \\
&\quad + (\partial_t B_2 + \partial_1 E_3 - \partial_3 E_1)\; dt \wedge dz \wedge dx \\
&\quad + (\partial_t B_3 + +\partial_1 E_2 - \partial_2 E_1)\; dt \wedge dx \wedge dy \\
&\quad + (\partial_1 B_1 + \partial_2 B_2 + \partial_3 B_3)\; dx \wedge dy \wedge dz \\
&= (\partial_t B_1 + (\boldsymbol{\nabla} \times \boldsymbol{E})_1)\; dt \wedge dy \wedge dz \\
&\quad + (\partial_t B_2 + (\boldsymbol{\nabla} \times \boldsymbol{E})_2)\; dt \wedge dz \wedge dx \\
&\quad + (\partial_t B_3 + (\boldsymbol{\nabla} \times \boldsymbol{E})_3)\; dt \wedge dx \wedge dy \\
&\quad + (\boldsymbol{\nabla} \cdot \boldsymbol{B})\; dx \wedge dy \wedge dz\,. \qquad (18.9)
\end{aligned}$$

Thus two of Maxwell's equations are equivalent to $dF = 0$.

For the other two equations we need $\star F$. Using the formula (17.11) for dual basis forms, it is easy to calculate that

$$\begin{aligned}
\star(dx \wedge dy) &= dt \wedge dz\,, \quad \star(dy \wedge dz) = dt \wedge dx\,, \quad \star(dz \wedge dx) = dt \wedge dy\,, \\
\star(dx \wedge dt) &= dy \wedge dz\,, \quad \star(dy \wedge dt) = dz \wedge dx\,, \quad \star(dz \wedge dt) = dx \wedge dy\,. \\
&\qquad\qquad (18.10)
\end{aligned}$$

We use these to calculate

$$\begin{aligned}
\star F &= \star(B + E \wedge dt) \\
&= B_1 dt \wedge dx + B_2 dt \wedge dy + B_3 dt \wedge dz \\
&\quad + E_1 dy \wedge dz + E_2 dz \wedge dx + E_3 dx \wedge dy\,. \qquad (18.11)
\end{aligned}$$

Then in the same way as for the previous calculation, we find

$$\begin{aligned} d{\star}F = & \, (\boldsymbol{\nabla}\cdot\boldsymbol{E})\, dx\wedge dy\wedge dz \\ & + (\partial_t E_1 - (\boldsymbol{\nabla}\times\boldsymbol{B})_1)\; dt\wedge dy\wedge dz \\ & + (\partial_t E_2 - (\boldsymbol{\nabla}\times\boldsymbol{B})_2)\; dt\wedge dz\wedge dx \\ & + (\partial_t E_3 + (\boldsymbol{\nabla}\times\boldsymbol{B})_3)\; dt\wedge dx\wedge dy\,. \end{aligned} \tag{18.12}$$

We need to relate this to the charge-current.

Define the current four-vector as

$$j^\mu\partial_\mu = \rho\partial_t + j^1\partial_1 + j^2\partial_2 + j^3\partial_3\,. \tag{18.13}$$

Then there is a corresponding one-form $j_\mu dx^\mu$ with $j_\mu = g_{\mu\nu}j^\nu$. So in terms of components,

$$j_\mu dx^\mu = -\rho dt + j_1 dx^1 + j_2 dx^2 + j_3 dx^3\,. \tag{18.14}$$

Then using Eq. (17.11) it is easy to calculate that

$$\begin{aligned} \star j = & -\rho\, dx\wedge dy\wedge dz + j_1\, dt\wedge dy\wedge dz \\ & + j_2\, dt\wedge dz\wedge dx + j_3\, dt\wedge dx\wedge dy\,. \end{aligned} \tag{18.15}$$

Comparing this equation with Eq. (18.12) we find that the other two Maxwell equations can be written as

$$d{\star}F = -{\star}j\,. \tag{18.16}$$

Finally, using Eq. (17.18), we see that the action of electromagnetism can be written as

$$-\frac{1}{2}\int F\wedge{\star}F \tag{18.17}$$

This expression holds in both flat and curved spacetimes. For the latter, with local coordinates (t,x,y,z) we find

$$F\wedge{\star}F = (\boldsymbol{B}^2 - \boldsymbol{E}^2)\sqrt{-g}\, dt\wedge dx\wedge dy\wedge dz\,. \tag{18.18}$$