Chapter 16

Metric tensor

• A **metric** on a vector space V is a function $g : V \times V \to \mathbb{R}$ which is

i) bilinear:

$$g(av_1 + v_2, w) = ag(v_1, w) + g(v_2, w)$$
$$g(v, w_1 + aw_2) = g(v, w_1) + ag(v, w_2)\,, \qquad (16.1)$$

i.e., g is a (0,2) tensor;

ii) symmetric:

$$g(v, w) = g(w, v); \qquad (16.2)$$

iii) non-degenerate:

$$g(v, w) = 0 \qquad \forall w \qquad \Rightarrow v = 0\,. \qquad (16.3)$$

□

• If for some $v, w \neq 0$, we find that $g(v, w) = 0$, we say that v, w are **orthogonal**. □

• Given a metric g on V, we can always find an **orthonormal basis** $\{e_\mu\}$ such that $g(e_\mu, e_\nu) = 0$ if $\mu \neq \nu$ and ± 1 if $\mu = \nu$. □

• If the number of (+1)'s is p and the number of (−1)'s is q, we say that the metric has **signature** (p, q).

We have defined a metric for a vector space. We can generalize this definition to a manifold $\mathcal{M}$ by the following.

• A **metric** g on a manifold $\mathcal{M}$ is a (0, 2) tensor field such that if (v, w) are smooth vector fields, $g(v, w)$ is a smooth function on $\mathcal{M}$, and has the properties (16.1), (16.2) and (16.3) mentioned earlier. □

It is possible to show that smoothness implies that the signature is constant on any connected component of $\mathcal{M}$, and we will assume that it is constant on all of $\mathcal{M}$.

A vector space becomes related to its dual space by the metric. Given a vector space V with metric g, and vector v defines a linear map $g(v,\cdot) : V \to \mathbb{R}\,, w \mapsto g(v,w) \in \mathbb{R}$. Thus $g(v,\cdot) \in V^*$ where V^* is the dual space of V. But $g(v,\cdot)$ is itself linear in v, so the map $V \to V^*$ defined by $g(v,\cdot)$ is linear. Since g is non-degenerate, this map is an isomorphism. It then follows that on a manifold we can use the metric to define a linear isomorphism between vectors and 1-forms.

In a basis, the components of the metric are $g_{\mu\nu} = g(e_\mu\,, e_\nu)$. This is an $n \times n$ matrix in an n-dimensional manifold. We can thus write $g(v,w) = g_{\mu\nu}v^\mu w^\nu$ in terms of the components. Non-degeneracy implies that this matrix is invertible. Let $g^{\mu\nu}$ denote the inverse matrix. Then, by definition of an inverse matrix, we have

$$g_{\mu\nu}g^{\nu\lambda} = \delta^\lambda_\mu = g^{\lambda\nu}g_{\mu\nu}\,. \tag{16.4}$$

Then the linear isomorphism takes the following form.

i) If $v = v^\mu e_\mu$ is a vector field in a chart, and $\{\lambda^\mu\}$ is the dual basis to $\{e_\mu\}$,

$$g(v,\cdot) = v_\mu \lambda^\mu\,, \tag{16.5}$$

where $v_\mu = g_{\mu\nu}v^\nu$.

ii) If $A = A_\mu\lambda^\mu$ is a 1-form written in a basis $\{\lambda^\mu\}$, the corresponding vector field is $A^\mu e_\mu$, where $A^\mu = g^{\mu\nu}A_\nu$.

This is the isomorphism between vector fields and 1-forms. (We could of course define a similar isomorphism between vectors and covectors without referring to a manifold.) A similar isomorphism holds for tensors, e.g. in terms of components,

$$T^{\mu\nu} \longleftrightarrow T^\mu{}_\nu \longleftrightarrow T_\mu{}^\nu \longleftrightarrow T_{\mu\nu} \tag{16.6}$$

$$T^{\mu\nu\rho\cdots} \longleftrightarrow T^{\mu\nu}{}_\rho{}^{\cdots} \longleftrightarrow T^{\mu\nu}{}_{\rho\cdots} \longleftrightarrow T_{\mu\nu\rho}{}^{\cdots} \longleftrightarrow \cdots \tag{16.7}$$

These correspondences are not equalities — the components are not equal. What it means is that, if we know one set of components, say $T^{\mu\nu\rho\cdots}$, and the metric, we also know every other set of components.

• Using the fact that a non-degenerate metric defines a 1-1 linear map between vectors and 1-forms, we can define an **inner product of 1−forms**, by

$$\langle A \mid B\rangle = g^{\mu\nu}A_\mu B_\nu \tag{16.8}$$

for 1-forms A, B . This result is independent of the choice of basis, i.e. independent of the coordinate system, just like the **inner product of vector fields**,

$$\langle v \mid w\rangle = g(v,w) = g_{\mu\nu}v^\mu w^\nu\,. \tag{16.9}$$

□

Given a manifold with metric, there is a canonical volume form dV (sometimes written as vol) , which in a coordinate chart reads

$$dV = \sqrt{|\det g_{\mu\nu}|}dx^1\wedge\cdots\wedge dx^n\,. \tag{16.10}$$

Note that despite the notation, this is not a 1-form, nor the gradient of some function V . This is clearly a volume form because it is an n-form which is non-zero everywhere, as $g_{\mu\nu}$ is non-degenerate.

We need to show that this definition is independent of the chart. Take an overlapping chart. Then in the new chart, the corresponding volume form is

$$dV' = \sqrt{|\det g'_{\mu\nu}|}dx'^1\wedge\cdots\wedge dx'^n\,. \tag{16.11}$$

We wish to show that $dV' = dV$. In the overlap,

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}dx^\nu = A^\mu_\nu dx^\nu \text{ (say)} \tag{16.12}$$

Then $dx'^1\wedge\cdots\wedge dx'^n = (\det A)dx^1\wedge\cdots\wedge dx^n$.

On the other hand, if we look at the components of the metric tensor in the new chart,

$$\begin{aligned} g'_{\mu\nu} &= g(\partial'_\mu, \partial'_\nu) \\ &= \left(\frac{\partial x^\alpha}{\partial x'^\mu}\,\partial_\alpha\,,\,\frac{\partial x^\beta}{\partial x'^\nu}\,\partial_\beta\right) \\ &= g\left(\left(A^{-1}\right)^\alpha_\mu\partial_\alpha\,,\left(A^{-1}\right)^\beta_\nu\partial_\beta\right) \\ &= \left(A^{-1}\right)^\alpha_\mu\left(A^{-1}\right)^\beta_\nu g_{\alpha\beta}\,. \end{aligned} \tag{16.13}$$

Taking determinants, we find

$$\det g'_{\mu\nu} = (\det A)^{-2} (\det g_{\mu\nu}) \ . \tag{16.14}$$

Thus

$$\sqrt{|\det g'_{\mu\nu}|} = |\det A|^{-1} \sqrt{|\det g_{\mu\nu}|} \,, \tag{16.15}$$

and so $dV' = dV$.

- This is called the **metric volume form** and written as

$$dV = \sqrt{|g|} dx^1 \wedge \cdots \wedge dx^n \tag{16.16}$$

in a chart. □

When we write dV , sometimes we mean the n-form as defined above, and sometimes we mean $\sqrt{|g|} d^n x$, the measure for the usual integral. Another way of writing the volume form in a chart is in terms of its components,

$$dV = \frac{\sqrt{|g|}}{n!} \epsilon_{\mu_1 \cdots \mu_n} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_n} \tag{16.17}$$

where ϵ is the totally antisymmetric Levi-Civita symbol, with $\epsilon_{12\cdots n} = +1$. Thus $\sqrt{|g|}\, \epsilon_{\mu_1 \cdots \mu_n}$ are the components of the volume form.