Chapter 14

Exterior derivative

The ***exterior derivative*** is a generalization of the gradient of a function. It is a map from p-forms to $(p+1)$-forms. This should be a derivation, so it should be linear,

$$d(\alpha + \omega) = d\alpha + d\omega \qquad \forall p\text{-forms}\, \alpha\,, \omega\,. \tag{14.1}$$

This should also satisfy Leibniz rule, but the algebra of p-forms is not a commutative algebra but a ***graded commutator*** algebra, i.e., involves a factor of $(-1)^{pq}$ for exchanges,

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha\,, \tag{14.2}$$

as we have seen. We wish to define the exterior derivative so that it is compatible with this property, i.e.,

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{pq} d\beta \wedge \alpha\,. \tag{14.3}$$

Alternatively we can write

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{p} \alpha \wedge d\beta\,. \tag{14.4}$$

This will be the Leibniz rule for wedge products. Note that it gives the correct result when one or both of α, β are 0-forms, i.e., functions. The two formulas are identical by virtue of the fact that $d\beta$ is a $(q+1)$-form, so that

$$\alpha \wedge d\beta = (-1)^{p(q+1)} d\beta \wedge \alpha\,. \tag{14.5}$$

We will try to define the exterior derivative in a way such that it has these properties.

Let us define the exterior derivative of a p-form ω in a chart as

$$d\omega = \frac{1}{p!}\,\partial_i\omega_{i_1\cdots i_p}\,dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \tag{14.6}$$

This clearly has the first property of linearity. To check the (graded) Leibniz rule, let us write $\alpha \wedge \beta$ in components. Then

$$\begin{aligned} d(\alpha\wedge\beta) &= \frac{1}{p!q!}\partial_i\left(\alpha_{i_1\cdots i_p}\beta_{j_1\cdots j_q}\right)\,dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{j_q} \\ &= \frac{1}{p!q!}\,\left[\left(\partial_i\alpha_{i_1\cdots i_p}\right)\beta_{j_1\cdots j_q} + \alpha_{i_1\cdots i_p}\left(\partial_i\beta_{j_1\cdots j_q}\right)\right]\,dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{j_q} \\ &= \frac{1}{p!q!}\,\left(\partial_i\alpha_{i_1\cdots i_p}\right)\beta_{j_1\cdots j_q}\,dx^i\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^{j_1}\wedge\cdots dx^{j_q} \\ &\qquad + \frac{1}{p!q!}\,(-1)^p\,\alpha_{i_1\cdots i_p}\left(\partial_i\beta_{j_1\cdots j_q}\right)\,dx^{i_1}\wedge\cdots\wedge dx^{i_p}\wedge dx^i\wedge dx^{j_1}\wedge\cdots dx^{j_q} \\ &= d\alpha\wedge\beta + (-1)^p\alpha\wedge d\beta\,. \end{aligned} \tag{14.7}$$

A third property of the exterior derivative immediately follows from here,

$$d^2 = 0\,. \tag{14.8}$$

To see this, we write

$$\begin{aligned} d(d\omega) &= \frac{1}{p!}\,d\left(\partial_i\omega_{i_1\cdots i_p}dx^i\wedge dx^{i_1}\wedge\cdots dx^{i_p}\right) \\ &= \frac{1}{p!}\,\partial_j\partial_i\omega_{i_1\cdots i_p}dx^j\wedge dx^i\wedge dx^{i_1}\wedge\cdots dx^{i_p}\,. \end{aligned} \tag{14.9}$$

But the wedge product is antisymmetric, $dx^j \wedge dx^i = -dx^i \wedge dx^j$, and the indices are summed over, so the above object must be antisymmetric in $\partial_j\,,\partial_i$. But that vanishes. So $d^2 = 0$ on all forms.

Note that we can also write

$$d\omega = \frac{1}{p!}\,\left(d\omega_{i_1\cdots i_p}\right)\wedge dx^{i_1}\wedge\cdots dx^{i_p}\,, \tag{14.10}$$

where the object in parentheses is a gradient 1-form corresponding to the gradient of the component.

Consider a 1-form $A = A_\mu dx^\mu$ where A_μ are smooth functions on $\mathcal{M}$. Then using this definition we can write

$$\begin{aligned} dA &= (dA_\nu)\wedge dx^\nu \\ &= \partial_\mu A_\nu dx^\mu\wedge dx^\nu \\ &= \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)dx^\mu\wedge dx^\nu \\ \Rightarrow\qquad (dA)_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu\,. \end{aligned} \tag{14.11}$$

We can generalize this result to write for a p-form,

$$\alpha = \frac{1}{p!}\alpha_{\mu_1\cdots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \tag{14.12}$$

$$\begin{aligned} d\alpha &= \frac{1}{p!}\left(d\alpha_{\mu_1\cdots\mu_p}\right)dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \\ &= \frac{1}{(p+1)!}\partial_{[\mu}\alpha_{\mu_1\cdots\mu_p]}dx^{\mu}\wedge dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p} \end{aligned}$$

$$\Rightarrow \qquad (d\alpha)_{\mu\mu_1\cdots\mu_p} = \partial_{[\mu}\alpha_{\mu_1\cdots\mu_p]} \tag{14.13}$$

Example: For $p=1$ i.e. for a 1-form A we get from this formula $(dA)_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, in agreement with our previous calculation.

For $p=2$ we have a 2-form, call it α. Then using this formula we get

$$\begin{aligned} (d\alpha)_{\mu\nu\lambda} &= \partial_{[\mu}\alpha_{\nu\lambda]} \\ &= \partial_\mu\alpha_{\nu\lambda} - \partial_\nu\alpha_{\mu\lambda} + \partial_\nu\alpha_{\lambda\mu} - \partial_\lambda\alpha_{\nu\mu} + \partial_\lambda\alpha_{\mu\nu} - \partial_\mu\alpha_{\lambda\nu}\,. \end{aligned} \tag{14.14}$$

Note that d is not defined on arbitrary tensors, but only on forms. □

By definition, $d^2 = 0$ on any p-form. So if $\alpha = d\beta$, it follows that $d\alpha = 0$. But given a p-form α for which $d\alpha = 0$, can we say that there must be some $(p-1)$-form β such that $\alpha = d\beta$?

• This is a good place to introduce some terminology. Any form ω such that $d\omega = 0$ is called **closed**, whereas any form α such that $\alpha = d\beta$ is called ***exact***. □

So every exact form is closed. Is every closed form exact? The answer is yes, in a sufficiently small neighbourhood. We say that every closed form is locally exact. Note that if a p-form $\alpha = d\beta$, we cannot uniquely specify the $(p-1)$-form β since for any $(p-2)$-form γ , we can always write $\alpha = d\beta'$, where $\beta' = \beta + d\gamma$.

Thus a more precise statement is that given any p-form α such that $d\alpha = 0$ in a neighbourhood of some point P, there is some neighbourhood of this point and some $(p-1)$-form β such that $\alpha = d\beta$ in that neighbourhood. But this may not be true globally. This statement is known as the **Poincaré lemma**. □

Example: In $\mathbb{R}^2$ remove the origin. Consider the 1-form

$$\alpha = \frac{xdy - ydx}{x^2+y^2}\,. \tag{14.15}$$

Then

$$\begin{aligned} d\alpha &= \left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2}\right) dx\wedge dy - \left(\frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2}\right) dy\wedge dx \\ &= \frac{2}{x^2+y^2}\, dx\wedge dy - 2\frac{x^2+y^2}{(x^2+y^2)^2}\, dx\wedge dy = 0\,. \end{aligned} \tag{14.16}$$

Introduce polar coordinates r, θ with $x = r\cos\theta\,, y = r\sin\theta\,.$ Then

$$dx = dr\cos\theta - r\sin\theta d\theta \qquad dy = dr\sin\theta + r\cos\theta d\theta$$

$$\begin{aligned} \alpha &= \frac{r\cos\theta\,(\sin\theta dr + r\cos\theta d\theta)}{r^2} - \frac{r\sin\theta\,(\cos\theta dr - r\sin\theta d\theta)}{r^2} \\ &= \frac{r^2\left(\cos^2\theta + \sin^2\theta\right) d\theta}{r^2} = d\theta\,. \end{aligned} \tag{14.17}$$

Thus α is exact, but θ is multivalued so there is no function f such that $\alpha = df$ everywhere. In other words, $\alpha = d\theta$ is exact only in a neighbourhood small enough that θ remains single-valued.