Chapter 13

Differential forms

There is a special class of tensor fields, which is so useful as to have a separate treatment. There are called **differential** $p-$**forms** or $p-$**forms** for short.

• A $p-$**form** is a $(0,p)$ tensor which is completely antisymmetric, i.e., given vector fields $v_1\,,\cdots,v_p\,,$

$$\omega\,(v_1\,,\cdots,v_i\,,\cdots,v_j\,,\cdots,v_p) = -\omega\,(v_1\,,\cdots,v_j\,,\cdots,v_i\,,\cdots,v_p) \tag{13.1}$$

for any pair $i,j\,.$ □

A 0-form is defined to be a function, i.e. an element of $C^\infty(\mathcal{M})\,,$ and a 1-form is as defined earlier.

The antisymmetry of any p-form implies that it will give a non-zero result only when the p vectors are linearly independent. On the other hand, no more than n vectors can be linearly independent in an n-dimensional manifold. So $p \leqslant n\,.$

Consider a 2-form $A\,.$ Given any two vector fields $v_1\,,v_2\,,$ we have $A(v_1\,,v_2) = -A(v_2\,,v_1)\,.$ Then the components of A in a chart are

$$A_{ij} = A\,(\partial_i\,,\partial_j) = -A_{ji}\,. \tag{13.2}$$

Similarly, for a p-form $\omega\,,$ the components are $\omega_{i_1\cdots i_p}\,,$ and components are multiplied by (-1) whenever any two indices are interchanged.

It follows that a p-form has $\binom{n}{p}$ independent components in n-dimensions.

Any 1-form produces a function when acting on a vector field. So given a pair of 1-forms A, B, it is possible to construct a 2-form ω

by defining

$$\omega(u,v) = A(u)B(v) - B(u)A(v), \qquad \forall u,v\,. \tag{13.3}$$

• This is usually written as $\omega = A\otimes B - B\otimes A$, where $\otimes$ is called the **outer product**. □

• Then the above construction defines a product written as

$$\omega = A\wedge B = -B\wedge A\,, \tag{13.4}$$

and called the ***wedge product***. Clearly, ω is a 2-form. □

Let us work in a coordinate basis, but the results we find can be generalized to any basis. The coordinate bases for the vector fields, $\{\partial_i\}$, and 1-forms, $\{dx^i\}$, satisfy $dx^i(\partial_j) = \delta^i_j$. A 1-form A can be written as $A = A_i dx^i$, and a vector field v can be written as $v = v^i\partial_i$, so that $A(v) = A_i v^i$. Then for the ω defined above and for any pair of vector fields u, v,

$$\begin{aligned} \omega(u,v) &= A(u)B(v) - B(u)A(v) \\ &= A_i u^i B_j v^j - B_i u^i A_j v^j \\ &= (A_iB_j - B_iA_j)\, u^i v^j\,. \end{aligned} \tag{13.5}$$

The components of ω are $\omega_{ij} = \omega(\partial_i\,,\partial_j)$, so that

$$\omega(u,v) = \omega(u^i\partial_i\,, v^j\partial_j) = \omega_{ij}u^iv^j\,. \tag{13.6}$$

Then $\omega_{ij} = A_iB_j - B_iA_j$ for the 2-form defined above. We can now construct a basis for 2-forms, which we write as $dx^i\wedge dx^j$,

$$dx^i\wedge dx^j = dx^i\otimes dx^j - dx^j\otimes dx^i\,. \tag{13.7}$$

Then a 2-form can be expanded in this basis as

$$\omega = \frac{1}{2!}\omega_{ij}dx^i\wedge dx^j\,, \tag{13.8}$$

because then

$$\begin{aligned} \omega(u,v) &= \frac{1}{2!}\omega_{ij}\left(dx^i\otimes dx^j - dx^j\otimes dx^i\right)(u,v) \\ &= \frac{1}{2!}\omega_{ij}\left(u^iv^j - u^jv^i\right) = \omega_{ij}u^iv^j\,. \end{aligned} \tag{13.9}$$

Similarly, a basis for $p-$forms is

$$dx^{i_1} \wedge \cdots \wedge dx^{i_p} = dx^{[i_1} \otimes \cdots \otimes dx^{i_p]}\,, \qquad (13.10)$$

where the square brackets stand for total antisymmetrization: all even permutations of the indices are added and all the odd permutations are subtracted. (Caution: some books define the 'square brackets' as antisymmetrization with a factor $1/p!$.) For example, for a 3-form, a basis is

$$\begin{aligned} dx^i \wedge dx^j \wedge dx^k = \; & dx^i \otimes dx^j \otimes dx^k - dx^j \otimes dx^i \otimes dx^k \\ & + dx^j \otimes dx^k \otimes dx^i - dx^k \otimes dx^j \otimes dx^i \\ & + dx^k \otimes dx^i \otimes dx^j - dx^i \otimes dx^k \otimes dx^j\,. \end{aligned} \qquad (13.11)$$

Then an arbitrary 3-form Ω can be written as

$$\Omega = \frac{1}{3!}\Omega_{ijk} dx^i \wedge dx^j \wedge dx^k\,. \qquad (13.12)$$

Note that there is a sum over indices, so that the factorial goes away if we write each basis 3-form up to permutations, i.e. treating different permutations as equivalent. Thus a $p-$form α can be written in terms of its components as

$$\alpha = \frac{1}{p!}\alpha_{i_1 \cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}\,. \qquad (13.13)$$

Examples: A 2-form in two dimensions can be written as

$$\begin{aligned} \omega &= \frac{1}{2!}\omega_{ij}\, dx^i \wedge dx^j \\ &= \frac{1}{2!}\left(\omega_{12} dx^1 \wedge dx^2 + \omega_{21} dx^2 \wedge dx^1\right) \\ &= \frac{1}{2!}\left(\omega_{12} - \omega_{21}\right) dx^1 \wedge dx^2 \\ &= \omega_{12}\, dx^1 \wedge dx^2\,. \end{aligned} \qquad (13.14)$$

□

A 2-form in three dimensions can be written as

$$\begin{aligned} \omega &= \frac{1}{2!}\omega_{ij} dx^i \wedge dx^j \\ &= \omega_{12}\, dx^1 \wedge dx^2 + \omega_{23}\, dx^2 \wedge dx^3 + \omega_{31}\, dx^3 \wedge dx^1 \end{aligned} \qquad (13.15)$$

□

In three dimensions, consider two 1-forms $\alpha = \alpha_i dx^i \,, \beta = \beta_i dx^i \,.$ Then

$$\begin{aligned}\alpha \wedge \beta = & \quad (\alpha_i\beta_j - \alpha_j\beta_i)\,\frac{1}{2!}\,dx^i \wedge dx^j \\ = & \quad \alpha_i\beta_j dx^i \wedge dx^j \\ = & \quad (\alpha_1\beta_2 - \alpha_2\beta_1)\,dx^1 \wedge dx^2 \\ & + (\alpha_2\beta_3 - \alpha_3\beta_2)\,dx^2 \wedge dx^3 \\ & + (\alpha_3\beta_1 - \alpha_1\beta_3)\,dx^3 \wedge dx^1 \,. \end{aligned} \tag{13.16}$$

The components are like the cross product of vectors in three dimensions. So we can think of the wedge product as a generalization of the cross product.

• We can also define the ***wedge product*** of a $p-$form α and a $q-$form β as a $(p+q)-$form satisfying, for any $p+q$ vector fields $v_1, \cdots, v_{p+q}\,,$

$$\alpha \wedge \beta\,(v_1, \cdots, v_{p+q}) = \frac{1}{p!q!}\sum_P (-1)^{\deg P}\alpha \otimes \beta\,(P\,(v_1, \cdots, v_{p+q}))\ . \tag{13.17}$$

Here P stands for a permutation of the vector fields, and $\deg P$ is 0 or 1 for even and odd permutations, respectively. In the outer product on the right hand side, α acts on the first p vector fields in a given permutation P, and β acts on the remaining q vector fields. □

The wedge product above can also be defined in terms of the components of α and β in a chart as follows.

$$\begin{aligned}\alpha &= \frac{1}{p!}\,\alpha_{i_1\cdots i_p}\,dx^{i_1} \wedge \cdots \wedge dx^{i_p} \\ \beta &= \frac{1}{q!}\,\beta_{j_1\cdots j_q}\,dx^{j_1} \wedge \cdots \wedge dx^{j_q} \\ \alpha \wedge \beta &= \frac{1}{p!q!}\,\alpha_{i_1\cdots i_p}\,\beta_{j_1\cdots j_q}\,\left(dx^{i_1} \wedge \cdots \wedge dx^{i_p}\right) \wedge \left(dx^{j_1} \wedge \cdots \wedge dx^{j_q}\right)\ . \end{aligned} \tag{13.18}$$

Note that $\alpha \wedge \beta = 0$ if $p+q > n\,,$ and that a term in which some i is equal to some j must vanish because of the antisymmetry of the wedge product.

It can be shown by explicit calculation that wedge products are associative,

$$\alpha \wedge (\beta \wedge \gamma) = (\alpha \wedge \beta) \wedge \gamma\,. \tag{13.19}$$

Cross-products are not associative, so there is a distinction between cross-products and wedge products. In fact, for 1-forms in three dimensions, the above equation is analogous to the identity for the triple product of vectors,

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} \,. \tag{13.20}$$

For a p-form α and q-form β, we find

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \,. \tag{13.21}$$

Proof: Consider the wedge product written in terms of the components. We can ignore the parentheses separating the basis forms since the wedge product is associative. Then we exchange the basis 1-forms. One exchange gives a factor of -1,

$$dx^{i_p} \wedge dx^{j_1} = -dx^{j_1} \wedge dx^{i_p} \,. \tag{13.22}$$

Continuing this process, we get

$$\begin{aligned} & dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q} \\ & \qquad = (-1)^p dx^{j_1} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_q} \\ & \qquad = \cdots \\ & \qquad = (-1)^{pq} dx^{j_1} \wedge \cdots \wedge dx^{j_q} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} \,. \end{aligned} \tag{13.23}$$

Putting back the components, we find

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha \tag{13.24}$$

as wanted. □

- The wedge product defines an algebra on the space of differential forms. It is called a ***graded commutative algebra***. □
- Given a vector field v, we can define its ***contraction*** with a p-form by

$$\iota_v \omega = \omega(v, \cdots) \tag{13.25}$$

with $p-1$ empty slots. This is a $(p-1)$-form. Note that the position of v only affects the sign of the contracted form. □

Example: Consider a 2-form made of the wedge product of two 1-forms, $\omega = \lambda \wedge \mu = \lambda \otimes \mu - \mu \otimes \lambda$. Then contraction by v gives

$$\iota_v \omega = \omega(v, \bullet) = \lambda(v)\mu - \mu(v)\lambda = -\omega(\bullet, v) \,. \tag{13.26}$$

If we have a p-form $\omega = \frac{1}{p!}\omega_{i_1\cdots i_p}\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, its contraction with a vector field $v = v^i\partial_i$ is

$$\iota_v\omega = \frac{1}{(p-1)!}\,\omega_{ii_2\cdots i_p} v^i dx^{i_2} \wedge \cdots \wedge dx^{i_p} \,. \tag{13.27}$$

Note the sum over indices. To see how the factor becomes $\frac{1}{(p-1)!}$, we write the contraction as

$$\iota_v\omega = \frac{1}{p!}\,\omega_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\left(v^i\partial_i\right) \,. \tag{13.28}$$

Since the contraction is done in the first slot, so we consider the action of each basis 1-form dx^{i_k} on ∂_i by carrying dx^{i_k} to the first position and then writing a $\delta_i^{i_k}$. This gives a factor of (-1) for each exchange, but we get the same factor by rearranging the indices of ω , thus getting a $+1$ for each index. This leads to an overall factor of p .

• given a diffeomorphism $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$, the **pullback** of a 1-form λ (on $\mathcal{M}_2$) is $\varphi^*\lambda$, defined by

$$\varphi^*\lambda(v) = \lambda(\varphi_* v) \tag{13.29}$$

for any vector field v on $\mathcal{M}_1$. □

Then we can consider the pullback $\varphi^* dx^i$ of a basis 1-form dx^i . For a general 1-form $\lambda = \lambda_i dx^i$, we have $\varphi^*\lambda = \varphi^*(\lambda_i dx^i)$. But

$$\varphi^*\lambda(v) = \lambda(\varphi_* v) = \lambda_i\, dx^i(\varphi_* v) \,. \tag{13.30}$$

Now, $dx^i(\varphi_* v) = \varphi^* dx^i(v)$ and the thing on the right hand side is a function on $\mathcal{M}_1$, so we can write this as

$$\varphi^*\lambda(v) = (\varphi^*\lambda_i)\varphi^* dx^i(v) \,, \tag{13.31}$$

where $\varphi^*\lambda_i$ are now functions on $\mathcal{M}_1$, i.e.

$$(\varphi^*\lambda_i)\big|_P = \lambda_i\big|_{\varphi(P)} \tag{13.32}$$

So we can write $\varphi^*\lambda = (\varphi^*\lambda_i)\,\varphi^* dx^i$. For the wedge product of two 1-forms,

$$\begin{aligned}
\varphi^*(\lambda \wedge \mu)(u, v) &= (\lambda \wedge \mu)(\varphi_* u\,, \varphi_* v) \\
&= \lambda \otimes \mu(\varphi_* u\,, \varphi_* v) - \mu \otimes \lambda(\varphi_* u\,, \varphi_* v) \\
&= \lambda(\varphi_* u)\mu(\varphi_* v) - \mu(\varphi_* u)\lambda(\varphi_* v) \\
&= \varphi^*\lambda(u)\varphi^*\mu(v) - \varphi^*\mu(u)\varphi^*\lambda(v) \\
&= (\varphi^*\lambda \wedge \varphi^*\mu)(u\,, v) \,.
\end{aligned} \tag{13.33}$$

Since u, v are arbitrary vector fields it follows that

$$\begin{aligned} \varphi^*(\lambda \wedge \mu) &= \varphi^*\lambda \wedge \varphi^*\mu \\ \varphi^*(dx^i \wedge dx^j) &= \varphi^* dx^i \wedge \varphi dx^j \,. \end{aligned} \tag{13.34}$$

Since the wedge product is associative, we can write (by assuming an obvious generalization of the above formula)

$$\begin{aligned} \varphi^*\left(dx^i \wedge dx^j \wedge dx^k\right) &= \varphi^*\left(\left(dx^i \wedge dx^j\right) \wedge dx^k\right) \\ &= \varphi^*\left(dx^i \wedge dx^j\right) \wedge \varphi^* dx^k \\ &= \varphi^* dx^i \wedge \varphi^* dx^j \wedge \varphi^* dx^k \,, \end{aligned} \tag{13.35}$$

and we can continue this for any number of basis 1-forms. So for any p-form ω, let us define the pullback $\varphi^*\omega$ by

$$\varphi^*\omega(v_1 \,, \cdots, v_p) = \omega\left(\varphi_* v_1 \,, \cdots, \varphi_* v_p\right) \,, \tag{13.36}$$

and in terms of components, by

$$\varphi^*\omega = \frac{1}{p!}\left(\varphi^*\omega_{i_1 \cdots i_p}\right) \varphi^* dx^{i_1} \wedge \cdots \wedge dx^{i_p} \,. \tag{13.37}$$

We assumed above that the pullback of the wedge product of a 2-form and a 1-form is the wedge product of the pullbacks of the respective forms, but it is not necessary to make that assumption – it can be shown explicitly by taking three vector fields and following the arguments used earlier for the wedge product of two 1-forms.

Then for any p-form α and q-form β we can calculate from this that

$$\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta \,. \tag{13.38}$$

Thus pullbacks commute with (are distributive over) wedge products.