Chapter 10

Local flows

We met local flows and integral curves in Chapter 6. Given a vector field v, write its local flow as ϕ_t.

• The collection ϕ_t for $t < \epsilon$ (for some $\epsilon > 0$, or alternatively for $t < 1$) is a **one−parameter group of local diffeomorphisms**. □

Consider the vector field in a neighbourhood U of a point $Q \in \mathcal{M}$. Since $\phi_t : U \to \mathcal{M}, Q \mapsto \gamma_Q(t)$ is **local diffeomorphism**, i.e. diffeomorphism for sufficiently small values of t, we can use ϕ_t to push forward vector fields. At some point P we have the curve $\phi_t(P)$. We push forward a vector field at $t = \epsilon$ to $t = 0$ and compare with the vector field at $t = 0$.

We recall that for a map $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ the pullback of a function $f \in C^\infty(\mathcal{M}_2)$ is defined as

$$\varphi^* f = f \circ \varphi : \mathcal{M}_1 \to \mathbb{R}\,, \tag{10.1}$$

and $\varphi^* f \in C^\infty(\mathcal{M}_1)$ if φ is C^∞.

The pushforward of a vector v_P is defined by

$$\varphi_* v_P(f) = v_P(f \circ \varphi) = v_P(\varphi^* f) \tag{10.2}$$

$$v_P \in T_P\mathcal{M}_1\,, \qquad \varphi_* v_P \in T_{\varphi(P)}\mathcal{M}_2\,. \tag{10.3}$$

If φ is a diffeomorphism, we can define the pushforward of a vector field v by

$$\begin{aligned} \varphi_* v(f)\big|_{\varphi(P)} &= v\,(f \circ \varphi)\big|_P \\ \textit{i.e.} \qquad \varphi_* v(f)\big|_Q &= v\,(f \circ \varphi)\big|_{\varphi^{-1}Q} \\ &= v\,(\varphi^* f)\big|_{\varphi^{-1}Q}\,. \end{aligned} \tag{10.4}$$

We can rewrite this definition in several different ways,

$$\begin{aligned}(\varphi_* v)(f) &= v\,(f\circ\varphi)\circ\varphi^{-1} \\ &= (\varphi^{-1})^*\,(v\,(f\circ\varphi)) \\ &= (\varphi^{-1})^*\,(v\,(\varphi^* f))\ . \end{aligned} \tag{10.5}$$

• If $\varphi:\mathcal{M}_1\to\mathcal{M}_2$ is not invertible, $\varphi_* v$ is not a vector field on $\mathcal{M}_2$. If φ^{-1} exists but is not differentiable, $\varphi_* v$ is not differentiable. But there are some φ and some v such that $\varphi_* v$ is a differentiable vector field, even if φ is not invertible or φ^{-1} is not differentiable. Then v and $\varphi_* v$ are said to be $\varphi-$***related***. □

Proposition: Given a diffeomorphism $\varphi:\mathcal{M}_1\to\mathcal{M}_2$ (say both C^∞ manifolds) the pushforward φ_* is an isomorphism on the Lie algebra of vector fields, i.e.

$$\varphi_*[u\,,v] = [\varphi_* u\,,\varphi_* v]\,. \tag{10.6}$$

Proof:

$$\begin{aligned}\varphi_*[u\,,v](f) &= [u\,,v]\,(f\circ\varphi)\circ\varphi^{-1} \\ &= u\,(v\,(f\circ\varphi))\circ\varphi^{-1} - u\leftrightarrow v\,, \end{aligned} \tag{10.7}$$

while

$$\begin{aligned}[\varphi_* u\,,\varphi_* v](f) &= \varphi_* u\,(\varphi_* v\,(f)) - u\leftrightarrow v \\ &= u\,(\varphi_* v\,(f)\circ\varphi)\circ\varphi^{-1} - u\leftrightarrow v \\ &= u\,\left(\left(v\,(f\circ\varphi)\circ\varphi^{-1}\right)\circ\varphi\right)\circ\varphi^{-1} - u\leftrightarrow v \\ &= u\,(v\,(f\circ\varphi))\circ\varphi^{-1} - u\leftrightarrow v\,. \end{aligned} \tag{10.8}$$

□

• A vector field v is said to be ***invariant*** under a diffeomorphism $\varphi:\mathcal{M}\to\mathcal{M}$ if $\varphi_* v = v$, i.e. if $\varphi_*(v_{P}) = v_{\varphi(P)}$ for all $P\in\mathcal{M}$. □

We can write for any $f\in C^\infty(\mathcal{M})$

$$\begin{aligned} (\varphi_* v)\,(f) &= \left(\varphi^{-1}\right)^*(v\,(\varphi^* f)) \\ \Rightarrow \qquad \varphi^*\left((\varphi_* v)\,(f)\right) &= v\,(\varphi^* f)\ , \\ \Rightarrow \qquad \varphi^*\circ\varphi_* v &= v\circ\varphi^*\,. \end{aligned} \tag{10.9}$$

So if v is an invariant vector field, we can write

$$\varphi^*\circ v = v\circ\varphi^*\,. \tag{10.10}$$

This expresses invariance under φ, and is satisfied by all differential operators invariant under φ.

Consider a vector field u, and the local flow (or one-parameter diffeomorphism group) ϕ_t corresponding to u,

$$\phi_t(Q) = \gamma_Q(t)\,, \qquad \dot{\gamma}_Q(t) = u(\gamma_Q(t))\,. \tag{10.11}$$

But for any $f \in C^\infty(\mathcal{M})$,

$$\begin{aligned}
\dot{\gamma}_Q(f) &= \frac{d}{dt}\left(f \circ \gamma_Q(t)\right) \\
&= \frac{d}{dt}\left(f \circ \phi_t(Q)\right) \\
&= \frac{d}{dt}\left(\phi_t^*(f)\right) = u_{\gamma_Q(t)}(f) \equiv u(f)\Big|_{\gamma_Q(t)}
\end{aligned} \tag{10.12}$$

At $t = 0$ we get the equation

$$\frac{d}{dt}\left(\phi_t^*(f)\right)\Big|_{t=0} = u(f)\Big|_Q \tag{10.13}$$

We can also write

$$\frac{d}{dt}\left(\phi_t^* f\right)(Q) = u(f)\left(\phi_t(Q)\right) = \phi_t^* u(f)(Q)\,. \tag{10.14}$$

This formula can be used to solve linear partial differential equations of the form

$$\frac{\partial}{\partial t} f\left(\boldsymbol{x}, t\right) = \sum_{i=1}^{n} v^i(\boldsymbol{x}) \frac{\partial}{\partial x^i} f(\boldsymbol{x}, t) \tag{10.15}$$

with initial condition $f(\boldsymbol{x}, 0) = g(\boldsymbol{x})$ and everything smooth. This is an equation on $\mathbb{R}^{n+1}$, so it can be on a chart for a manifold as well.

We can treat $v^i(\boldsymbol{x})$ as components of a vector field v. Then a solution to this equation is

$$\begin{aligned}
f(\boldsymbol{x}, t) &= \phi_t^* g(\boldsymbol{x}) \\
&\equiv g\left(\phi_t(\boldsymbol{x})\right) \equiv g \circ \phi_t(\boldsymbol{x})\,,
\end{aligned} \tag{10.16}$$

where ϕ_t is the flow of v.

Proof:

$$\frac{\partial}{\partial t} f(\boldsymbol{x}, t) = \frac{d}{dt}\left(\phi_t^* g\right) = v(f) \equiv v^i \frac{\partial f}{\partial x^i}\,, \tag{10.17}$$

using Eq. (10.13) . □

Thus the partial differential equation can be solved by finding the integral curves of v (the flow of v) and then by pushing (also called **dragging**) g along those curves. It can be shown, using well-known theorems about the uniqueness of solutions to first order partial differential equations, that this solution is also unique.

Example: Consider the equation in 2+1 dimensions

$$\frac{\partial}{\partial t} f(\boldsymbol{x}, t) = (x - y)\left(\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right) \tag{10.18}$$

with initial condition $f(\boldsymbol{x}, 0) = x^2 + y^2$. The corresponding vector field is $v(\boldsymbol{x}) = (x - y, -x + y)$. The integral curve passing through the point $P = (x_0, y_0)$ is given by the coordinates

$$\gamma(t) = (v_x(P)t + x_0, v_y(P)t + y_0)\,, \tag{10.19}$$

so the integral curve passing through (x, y) in our example is given by

$$\begin{aligned} \gamma(t) &= ((x - y)t + x, (-x + y)t + y) \\ &= \Phi_t(x, y)\,, \end{aligned} \tag{10.20}$$

the flow of v. So the solution is

$$\begin{aligned} f(\boldsymbol{x}, t) &= \Phi_t^* f(\boldsymbol{x}, 0) = f(\boldsymbol{x}, 0) \circ \Phi_t(x, y) \\ &= [(x - y)t + x]^2 + [(-x + y)t + y]^2 \\ &= (x - y)^2 t^2 + x^2 + 2(x - y)xt + (x - y)^2 t^2 + y^2 - 2(x - y)yt \\ &= 2(x - y)^2 t^2 + (x^2 + y^2)(1 + 2t) - 4xyt\,. \end{aligned} \tag{10.21}$$