Chapter 1

Topology

We start by defining a topological space.

• A **topological space** is a set $\mathcal{S}$ together with a collection $\mathscr{O}$ of subsets called **open sets** such that the following are true:

i) the empty set $\emptyset$ and $\mathcal{S}$ are open, $\emptyset, \mathcal{S} \in \mathscr{O}$

ii) the intersection of a *finite* number of open sets is open; if $U_1, U_2 \in \mathscr{O}$, then $U_1 \cap U_2 \in \mathscr{O}$

iii) the union of any number of open sets is open, if $U_i \in \mathscr{O}$, then $\bigcup_i U_i \in \mathscr{O}$ irrespective of the range of i. □

It is the pair $\{\mathcal{S}, \mathscr{O}\}$ which is, precisely speaking, a topological space, or a **space with topology**. But it is common to refer to $\mathcal{S}$ as a topological space which has been given a **topology** by specifying $\mathscr{O}$.

Example: $\mathcal{S} = \mathbb{R}$, the real line, with the open sets being open intervals $]a, b[$, i.e. the sets $\{x \in \mathbb{R} \,|\, a < x < b\}$ and their unions, plus $\emptyset$ and $\mathbb{R}$ itself. Then (i) above is true by definition.

For two such open sets $U_1 =]a_1, b_1[$ and $U_2 =]a_2, b_2[$, we can suppose $a_1 < a_2$. Then if $b_1 \leqslant a_2$, the intersection $U_1 \cap U_2 = \emptyset \in \mathscr{O}$. Otherwise $U_1 \cap U_2 =]a_2, b_1[$ which is an open interval and thus $U_1 \cap U_2 \in \mathscr{O}$. So (ii) is true.

And (iii) is also true by definition. □

Similarly $\mathbb{R}^n$ can be given a topology via open rectangles, i.e. via the sets $\{(x_1, \cdots, x_n) \in \mathbb{R}^n \,|\, a_i < x_i < b_i\}$. This is called the **standard** or usual topology of $\mathbb{R}^n$.

• The **trivial** topology on $\mathcal{S}$ consists of $\mathscr{O} = \{\emptyset, \mathcal{S}\}$. □

• The **discrete** topology on a set $\mathcal{S}$ is defined by $\mathscr{O} = \{A \mid A \subset \mathcal{S}\}$, i.e., $\mathscr{O}$ consists of all subsets of $\mathcal{S}$. □

• A set A is **closed** if its complement in $\mathcal{S}$, also written $\mathcal{S}\backslash A$ or as $A^{\complement}$, is open. □

Closed rectangles in $\mathbb{R}^n$ are closed sets as are closed balls and single point sets.

A set can be neither open nor closed, or both open and closed. In a discrete topology, every set $A \subset \mathcal{S}$ is both open and closed, whereas in a trivial topology, any set $A \neq \emptyset$ or $\mathcal{S}$ is neither open nor closed.

The collection $\mathscr{C}$ of closed sets in a topological space $\mathcal{S}$ satisfy the following:

i) the empty set $\emptyset$ and $\mathcal{S}$ are closed, $\emptyset, \mathcal{S} \in \mathscr{C}$

ii) the union of a *finite* number of closed sets is closed; if $A_1, A_2 \in \mathscr{C}$, then $A_1 \cup A_2 \in \mathscr{C}$

iii) the intersection of any number of closed sets is closed, if $A_i \in \mathscr{C}$, then $\bigcap_i A_i \in \mathscr{C}$ irrespective of the range of i.

Closed sets can also be used to define a topology. Given a set $\mathcal{S}$ with a collection $\mathscr{C}$ of subsets satisfying the above three properties of closed sets, we can always define a topology, since the complements of closed sets are open. (Exercise!)

• An **open neighbourhood** of a point P in a topological space $\mathcal{S}$ is an open set containing P. A **neighbourhood** of P is a set containing an open neighbourhood of P. Neighbourhoods can be defined for sets as well in a similar fashion. □

Examples: For a point $x \in \mathbb{R}$, and for any $\epsilon > 0$,
$]x - \epsilon, x + \epsilon[$ is an open neighbourhood of x,
$[x - \epsilon, x + \epsilon[$ is a neighbourhood of x,
$\{x - \epsilon \leqslant y < \infty\}$ is a neighbourhood of x,
$[x, x + \epsilon[$ is not a neighbourhood of x. □

• A topological space is **Hausdorff** if two distinct points have disjoint neighbourhoods. □

Topology is useful to us in defining continuity of maps.

• A map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is **continuous** if given any open set $U \subset \mathcal{S}_2$ its inverse image (or pre-image, what it is an image of) $f^{-1}(U) \subset \mathcal{S}_1$ is open. □

When this definition is applied to functions from $\mathbb{R}^m$ to $\mathbb{R}^n$, it is the same as the usual $\epsilon - \delta$ definition of continuity, which says that

• $f : \mathbb{R}^m \to \mathbb{R}^n$ is **continuous** at $\boldsymbol{x}_0$ if given $\epsilon > 0$, we can always find a $\delta > 0$ such that $|f(\boldsymbol{x}) - f(\boldsymbol{x}_0)| < \epsilon$ whenever $|\boldsymbol{x} - \boldsymbol{x}_0| < \delta$. □

For the case of functions from a topological space to $\mathbb{R}^n$, this definition says that

• $f : \mathcal{S} \to \mathbb{R}^n$ is **continuous** at $s_0 \in \mathcal{S}$ if given $\epsilon > 0$, we can always find an open neighbourhoood U of s_0 such that $|f(s) - f(s_0)| < \epsilon$ whenever $s \in U$. □

• If a map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is one-to-one and onto, i.e. a **bijection**, and both f and f^{-1} are continuous, f is called a **homeomorphism** and we say that $\mathcal{S}_1$ and cs_2 are **homeomorphic**. □

Proposition: The composition of two continuous maps is a continuous map.

Proof: If $f : \mathcal{S}_1 \to \mathcal{S}_2$ and $g : \mathcal{S}_2 \to \mathcal{S}_3$ are continuous maps, and U is some open set in $\mathcal{S}_3$, then its pre-image $g^{-1}(U)$ is open in $\mathcal{S}_2$. So $f^{-1}(g^{-1}(U))$, which is the pre-image of that, is open in $\mathcal{S}_1$. Thus $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$ is open in $\mathcal{S}_1$. Thus $g \circ f$ is continuous. □